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LETTER TO THE EDITOR

On Sompolinsky's spin glass theory approached from Thouless–Anderson–Palmer equations

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Abstract. A derivation of Sompolinsky's result for the long range spin glass is given starting from the set of Thouless–Anderson–Palmer (TAP) equations. No replicas are used. To describe the system at successive levels of averaging (over successive time scales or associated TAP solutions) the method introduces an infinite sequence of fluctuating local fields.

Currently there exist three successful approaches to the solution of the long range spin glass proposed by Sherrington and Kirkpatrick (1975). The first is a static approach that starts from the spin Hamiltonian and searches for appropriate ansatzes of $q_{\alpha\beta}$, the order parameter in replica space, an approach heralded by Parisi's replica symmetry breaking solution (Parisi 1979). The second approach is the dynamical one of Sompolinsky and Zippelius (1981) where the block ansatz in replica space is replaced by an ansatz for a time scale hierarchy. Both give identical values for thermodynamic quantities (De Dominicis and Young 1983a, b, Sommers 1983a, b). It is still under discussion whether the theory describes equilibrium or non-equilibrium properties (Houghton *et al* 1983a, b, De Dominicis and Young 1983a, b, Parisi 1983a, b, Sommers 1983a, b, Sompolinsky and Zippelius 1983). In between lies the Thouless–Anderson–Palmer (TAP) (1977) mean field type approach and it is an interesting question to see how TAP equations fit in. These equations are known to possess a large number of solutions (Bray and Moore 1980a, b, De Dominicis *et al* 1980). In particular, to define an average over solutions one has to specify which weight is attributed to each solution. A canonical weight is known to render this TAP approach equivalent to the initial Hamiltonian scheme (De Dominicis and Young 1983a, b). At the end of this article and in separate work (De Dominicis *et al* 1983), we consider an average with white weight over an appropriate window of the solutions.

Since the solutions of low free energy are highly correlated (Bray and Moore 1980a, b), Dasgupta and Sompolinsky (1983) chose to introduce an *ad hoc* distribution of overlap between them (the analogue of the time scale hierarchy ansatz) that led again to the Sompolinsky free energy. In this paper we wish to follow a distinct way: we are going to write TAP equations for successive levels of description. These may be interpreted as successive averages over solutions that are accessible within successively increasing time scales.

The model is described by a Hamiltonian of N spins S_i interacting via random exchange J_{ij}

$$H = -\tfrac{1}{2} \sum_{i \neq j} J_{ij} S_i S_j - \sum_i h_i^{\mathrm{e}} S_i. \tag{1}$$

We consider the Ising version of the model ($S_i = \pm 1$); h_i^{e} are local external fields. The J_{ij} are independent Gaussian random variables,

$$\overline{J_{ij}} = 0, \qquad \overline{J_{ij}^2} = J^2/N. \tag{2}$$

The range of validity of the high temperature solution of the SK model is determined by the de Almeida–Thouless (1978) line in the h^{e}–T plane ($h_i^{\mathrm{e}} \equiv h^{\mathrm{e}}$). The first attempt to treat the problem beyond that is due to TAP. The TAP equations are mean field type equations for local magnetisations m_i ($\beta = 1/T$):

$$\tanh^{-1} m_i = \beta h_i^{\mathrm{e}} + \sum_j \beta J_{ij} m_j - (\beta J)^2 (1 - q_{\mathrm{EA}}) m_i. \tag{3}$$

The local field is a sum of external field, Weiss mean field and Onsager reaction field. The last term is due to the fact that up to the desired order in $1/N$ the effect of spin S_i onto m_j has to be subtracted from the local field acting on S_i. Here

$$q_{\mathrm{EA}} = N^{-1} \sum_i \overline{m_i^2} \tag{4}$$

is the Edwards–Anderson order parameter.

Let us rewrite TAP in terms of local fields

$$h_i = \sum_j (J_{ij} - \mu \delta_{ij}) m(h_j) \tag{5}$$

with

$$\mu = \beta J^2 (1 - q_{\mathrm{EA}}) \qquad \text{and} \qquad m(h_j) = \tanh(\beta(h^{\mathrm{e}} + h_j)).$$

One may then compute (Sommers 1978) the probability distribution of the local field h_i. If one assumes a single solution for (5) one is led to the Sommers solution involving the two parameters q_{EA} and the anomaly

$$\Delta = N^{-1} \sum_j \overline{\partial m_j / \partial \beta h_j^{\mathrm{e}}} - (1 - q_{\mathrm{EA}}). \tag{6}$$

This solution is known to be unstable in the Hamiltonian approach (De Dominicis and Garel 1979). From a dynamical point of view it remains stable for short enough times but eventually decays (Khurana 1983, Sommers 1983a, b, Sompolinsky and Zippelius 1983). The time scale hierarchical ansatz suggests to replace (5) by a sequence of equations that depend on those parts of the local field which fluctuate on the corresponding time scale. Namely we write

$$\begin{aligned} h_i^{(0)} &= \sum_j (J_{ij} - \mu \delta_{ij}) m^{(0)}(h_j^{(0)}), \\ h_i^{(1)} &= \sum_j (J_{ij} - \mu \delta_{ij}) (m^{(1)}(h_j^{(0)}, h_j^{(1)}) - m^{(0)}(h_j^{(0)})), \\ &\vdots \\ h_i^{(R)} &= \sum_j (J_{ij} - \mu \delta_{ij}) (m^{(R)}(h_j^{(0)}, \ldots, h_j^{(R)}) - m^{(R-1)}(h_j^{(0)}, \ldots, h_j^{(R-1)})), \end{aligned} \tag{7}$$

with the requirement

$$m^{(R)}(h_j^{(0)}, \ldots, h_j^{(R)}) = \tanh(\beta(h^e + h_j^{(0)} + \ldots + h_j^{(R)}) \tag{8}$$

for $R \to \infty$.

At this stage we assume that the functions $m^{(0)}$, $m^{(1)}$, $m^{(2)}, \ldots$ are given and that the set of equations (7) is uniquely soluble. Then we can proceed to calculate the *probability distribution of the local fields* $\{h_i^\alpha\}$.

It is convenient to consider a *more general* set of equations

$$h_i^\alpha = \sum_j (J_{ij} - \mu\delta_{ij}) M_j^\alpha \tag{9}$$

where M_j^α are functions of the local fields h_j^β. The probability distribution of h_i^α is obtained by averaging over the bond distribution

$$\begin{aligned} Z_N &= \int \prod_{i\alpha} \left[\mathrm{d}h_i^\alpha\, \delta\left(h_i^\alpha - \sum_j (J_{ij} - \mu\delta_{ij}) M_j^\alpha \right) \right] \det\left(\delta_{ij}\delta^{\alpha\beta} - (J_{ij} - \mu\delta_{ij}) \frac{\partial M_j^\alpha}{\partial h_j^\beta} \right) \\ &= \int \prod_{i\alpha} \left\{ \mathrm{d}h_i^\alpha \frac{\mathrm{d}\hat{m}_i^\alpha}{2\pi} \mathrm{d}\eta_i^{+\alpha}\, \mathrm{d}\eta_i^\alpha \exp\left[-i\hat{m}_i^\alpha \left(h_i^\alpha - \sum_j (J_{ij} - \mu\delta_{ij}) M_j^\alpha \right) \right.\right. \\ &\qquad \left.\left. + \eta_i^{+\alpha} \left(\eta_i^\alpha - \sum_j (J_{ij} - \mu\delta_{ij}) \partial M_j^\alpha \right) \right] \right\} \end{aligned} \tag{10}$$

where we have introduced a constraint variable $\hat{m}_i^\alpha$ and anticommuting variables $\eta_i^{+\alpha}$, η_i^α. We further introduced the abbreviation

$$\partial M_j^\alpha = \sum_\beta (\partial M_j^\alpha / \partial h_j^\beta) \eta_j^\beta. \tag{11}$$

Now we can average Z_N with respect to J_{ij}:

$$\begin{aligned} \overline{Z_N} &= \int \prod_{i\alpha} \left(\mathrm{d}h_i^\alpha \frac{\mathrm{d}\hat{m}_i^\alpha}{2\pi} \mathrm{d}\eta_i^{+\alpha}\, \mathrm{d}\eta_i^\alpha \exp[-\mathrm{i}\hat{m}_i^\alpha (h_i^\alpha + \mu M_i^\alpha) + \eta_i^{+\alpha}(\eta_i^\alpha + \mu\partial M_i^\alpha)] \right) \\ &\quad \times \exp\left(-\frac{J^2}{2N} \sum_{\alpha\beta} \sum_{ij} (\hat{m}_i^\alpha \hat{m}_i^\beta M_j^\alpha M_j^\beta + \hat{m}_i^\alpha M_i^\beta \hat{m}_j^\beta M_j^\alpha \right. \\ &\qquad \left. + \eta_i^{+\alpha} \partial M_i^\beta \eta_j^{-\beta} \partial M_j^\alpha) \right) \end{aligned} \tag{12}$$

where we have neglected terms which lead to O(1) corrections inside the integral for large N. We can decouple, as usual, the last terms by the Hubbard–Stratonovich variables and perform a saddle point integration for large N which means that we consistently keep at the mean field level. Then all sites are effectively decoupled and we can forget about site indices. We get the *mean field equations*

$$\begin{aligned} q^{\alpha\beta} &= \langle M^\alpha M^\beta \rangle, & \hat{q}^{\alpha\beta} &= -\langle \hat{m}^\alpha \hat{m}^\beta \rangle, \\ g^{\alpha\beta} &= \langle \mathrm{i}\hat{m}^\alpha M^\beta \rangle, & n^{\alpha\beta} &= \langle \eta^{+\alpha} \partial M^\beta \rangle, \end{aligned} \tag{13}$$

where the brackets are defined with the weight factor in the normalisation integral

$$\begin{aligned} Z = \int \prod_\alpha \left\{ \mathrm{d}h^\alpha \frac{\mathrm{d}\hat{m}^\alpha}{2\pi} \mathrm{d}\eta^{+\alpha}\, \mathrm{d}\eta^\alpha \exp\left[-i\hat{m}^\alpha \left(h^\alpha + \mu M^\alpha - J^2 \sum_\beta g^{\beta\alpha} M^\beta \right) \right.\right. \\ \left.\left. + \eta^{+\alpha} \left(\eta^\alpha + \mu\partial M^\alpha - J^2 \sum_\beta g^{\beta\alpha} \partial M^\beta \right) - \frac{J^2}{2} \sum_\beta q^{\alpha\beta} \hat{m}^\alpha \hat{m}^\beta \right] \right\}. \end{aligned} \tag{14}$$

Consider the solution

$$\hat{q}^{\alpha\beta}=0, \qquad n^{\alpha\beta}=g^{\alpha\beta}, \tag{15}$$

which obviously exists with the weight given by (14). The Jacobian, implicit from the η^+, η integration, allows us to introduce the *independent Gaussian variables*

$$\xi^\alpha = h^\alpha + \mu M^\alpha - J^2 \sum_\beta g^{\beta\alpha} M^\beta \tag{16}$$

with the properties

$$\langle \xi^\alpha \rangle = 0, \qquad \langle \xi^\alpha \xi^\beta \rangle = J^2 q^{\alpha\beta}, \tag{17}$$

$$\langle (\partial/\partial\xi^\alpha) M^\beta \rangle = g^{\alpha\beta}. \tag{18}$$

We see immediately that Z is normalised to 1. A full consistency proof would require a stability analysis. Note the formal analogy to the time dependent formulation of Sompolinsky and Zippelius.

Let us now go back to equation (7) with

$$M^\alpha = m^\alpha(h^{(0)}, h^{(1)}, \ldots, h^\alpha) - m^{\alpha-1}(h^{(0)}, \ldots, h^{\alpha-1}). \tag{19}$$

We see then at once from (16) and (18) that $g^{\alpha\beta}$ admits generally a *triangular solution*:

$$g^{\alpha\beta} = 0 \qquad \text{for } \alpha > \beta. \tag{20}$$

Assuming for the time being that $m^{\alpha-1}$ is just the average of m^α with respect to ξ^α,

$$m^{\alpha-1}(h^{(0)}, \ldots, h^{\alpha-1}) = \overline{m^\alpha(h^{(0)}, \ldots, h^\alpha)}^{\xi^\alpha}, \tag{21}$$

we then consistently find $q^{\alpha\beta}$ and $g^{\alpha\beta}$ diagonal:

$$q^{\alpha\beta} = \delta^{\alpha\beta}(\langle (m^\alpha)^2 \rangle - \langle (m^{\alpha-1})^2 \rangle), \tag{22}$$

$$g^{\alpha\beta} = \langle \partial M^\beta / \partial \xi^\alpha \rangle = \delta^{\alpha\beta} \langle \partial M^\alpha / \partial \xi^\alpha \rangle. \tag{23}$$

This leads together with the asymptotic form of m^α (for $R \to \infty$), equation (8), in the continuous limit to Sompolinsky's solution (Sompolinsky 1981).

The choice of the functions m^α given by (21) seems to be rather arbitrary. It can *physically be justified* in the following way. Let us start from the TAP equations (5)

$$h_i^s = \sum_j (J_{ij} - \mu\delta_{ij}) \tanh(\beta(h^e + h_j^s)) \tag{24}$$

where we have attached a solution index s and for simplicity assumed that μ is solution independent. Let us assume that we have introduced some kind of dynamics and that for large but finite N the system transits for large time from one (almost) solution to another. Thus the time development for large times breaks (infinitesimally) the symmetry between solutions. With each solution index we can associate a time index as well. We measure the time scales by the number of solutions, T_0, $T_1, \ldots,$ which are accessible. We divide the largest time scale T_0 into blocks of size T_1, T_1 into blocks of size T_2 and so on, with the relation

$$T_0 \gg T_1 \gg T_2 \gg \ldots. \tag{25}$$

Thus each time index s can be written as $s = (\alpha_0, \alpha_1, \alpha_2, \ldots)$, indicating which block it belongs to on a certain scale ($\Sigma_{\alpha_i} = T_i / T_{i+1}$). We then may write the local field as a sum of contributions which average out on successive levels:

$$h_j^s = h_j^{(0)} + h_j^{\alpha_0} + h_j^{\alpha_0\alpha_1} + h_j^{\alpha_0\alpha_1\alpha_2} + \ldots. \tag{26}$$

Averaging the TAP equations (24) on successive levels we get

$$\begin{aligned} h_i^{(0)} &= \sum_j (J_{ij} - \mu\delta_{ij}) T_0^{-1} \sum_{\alpha_0'\alpha_1'\ldots} \tanh(\beta(h_j^e + h_j^{(0)} + h_j^{\alpha_0'} + h_j^{\alpha_0'\alpha_1'})), \\ h_i^{(0)} + h_i^{\alpha_0} &= \sum_j (J_{ij} - \mu\delta_{ij}) T_1^{-1} \sum_{\alpha_1'\ldots} \tanh(\beta(h_j^e + h_j^{(0)} + h_j^{\alpha_0} + h_j^{\alpha_0\alpha_1'} + \ldots)), \qquad (27) \\ &\vdots \end{aligned}$$

We see that we approximate in this way a certain solution h_j^s of the TAP equations by a sequence of the type we have already considered. In addition, calculating the joint probability of h_j^s, we can show that under condition (25) we can replace the sums over blocks on the right-hand side of (27) by averages over the resulting probability law just as we did in (21). Another way to get the same result is starting only from the last equation of (27) which is just (24). Assuming that the solution symmetry is broken in some infinitesimal way (in order to avoid replicas!) we can solve the resulting correlations $g^{ss'}$, $q^{ss'}$ by Parisi-like block structures of size T_0, $T_1, \ldots$. This leads to the same result as mentioned above, and corresponds to a (truncated) white average over solutions. Details will be published elsewhere.

To summarise, we have shown that starting from TAP equations, one may recover Sompolinsky's solution, provided one introduces to describe the system a sequence of fluctuating local fields. These local fields can be irrespectively associated with either time scales or with solutions accessible within those time scales. Sompolinsky's free energy is then written in terms of a sequence of magnetisations at an increasing level of averaging (over time scales or associated solutions) depending upon a decreasing number of fluctuating local fields.

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